

Random sets and intersections

Abstract The following class of problems arose out of vain attempts to show that the Pascal's triangle adic transformation has trivial spectrum. Partition a set of size N into sets of size $S \equiv S(N)$ (ignoring leftovers). What is the likelihood that a set of size $K \equiv K(N)$ will intersect each set in the partition in at least $R \equiv R(N)$ members (as N increases)? Via elementary techniques and under reasonable hypotheses, we obtain an easy-to-use formula. Although different from the corresponding minimum problem for balls and bins (with $m = K$ balls and $n = N/S$ bins), under modest constraints, the asymptotic probabilities are the same.

David Handelman¹

Let N be a (large) integer, and $S < N$, R , and K positive integers. Chop the set $\{1, 2, 3, \dots, N\}$ into $\lfloor N/S \rfloor$ disjoint intervals of length S (that is, $\{1, \dots, S\}$, $\{S+1, \dots, 2S\}$, etc), discarding the remainder. Regarding $S \equiv S(N)$, $R \equiv R(N)$, and $K \equiv K(N)$ as (given) functions of N , we want to determine the likelihood that a subset of $\{1, 2, 3, \dots, N\}$ with K elements will hit every one of the $\lfloor N/S \rfloor$ subintervals in at least R points (as $N \rightarrow \infty$). In order for the limiting probability to exist, there must be (mild) constraints on the variables, S, R, K, N .

Here $G_j(t) = t^j e^{-t}$.

THEOREM 1.1 The limiting probability that a subset of $\{1, 2, \dots, N\}$ of cardinality $K(N)$ will intersect each of the $\lfloor N/S(N) \rfloor$ disjoint sets of cardinality $S(N)$ in at least $R(N)$ members is e^{-c} , if

$$c = \lim_{N \rightarrow \infty} \frac{N \cdot G_{R-1}\left(\frac{SK}{N}\right)}{S \cdot (R-1)!}$$

exists in $\mathbf{R}^+ \cup \{\infty\}$, and provided that the following conditions hold.

- (a) $R(N)^2 = \mathbf{o}(S(N))$;
- (b) $NR(N) = \mathbf{o}(S(N)K(N))$;
- (c) $R(N)S(N) \vee R(N)K(N) = \mathbf{o}(N)$.

The method of proof is completely elementary, using inclusion/exclusion, Bonferroni's inequalities, and log concavity of various sequences.

Examples. It is convenient to pose the question in the form, given $R(N)$ and $S(N)$, how large must $K(N)$ be to guarantee occupation either with almost certainty, or with probability $1/e$.

The function G_j , given by $t \mapsto t^j e^{-t}$, if restricted to $t \geq j$, is decreasing, and its image on this interval is $(0, (j/e)^j)$. We may define its inverse function, $T_j : (0, (j/e)^j) \rightarrow (j, \infty)$. This is a relative of the Lambert W-function. It is decreasing, and on $(0, \epsilon)$, it expands as $T_j(t) \sim -\ln t + j \ln |\ln t| + O(\ln |\ln t| / |\ln t|)$.

Within the ranges permitted by (a–c), the probability of appropriately hitting the all the subsets is monotone increasing in $K(N)$ and decreasing in $R(N)$ and $N/S(N)$. Beyond this range, monotonicity is obvious for $R(N)$ and $N/S(N)$, but not clear for K , since we are calculating the relative probabilities with respect to the number of K -element sets.

EXAMPLES In these examples, we assume constraints (a–c) apply.

(a) If $R = 1$, consider $(N/S)e^{-SK/N}$. To obtain $c = t$ (with $0 < t < \infty$), we would have $K \sim (N/S)(\ln(N/S) - \ln t)$; to obtain $c = \infty$, we would have to have $KS/N - \ln(N/S) \rightarrow -\infty$ (in addition to the constraint that $SK/N \rightarrow \infty$). In particular, if $S(N) = \sqrt{N}$, then $c = 1$ arises when $K(N) \sim \frac{1}{2}\sqrt{N} \ln N$, and $c = \infty$ (practical certainty of unsuccessful occupation) occurs when $K/\sqrt{N} - \frac{1}{2} \ln N \rightarrow -\infty$.

(b) When $R = r + 1$ (constant in N), the situation is slightly more complicated. For all sufficiently large N (needed to ensure that SK/NR is big), $K(N) \sim (N/S)T_r(t \cdot r! \cdot S/N)$. As $S/N \rightarrow 0$, we can use the

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asymptotic expansion of T_r (for very small $s > 0$), $T_r(s) \sim -\ln s + r \ln |\ln s|$ (the next term is $\ln |\ln s|/|\ln s|$, so is irrelevant); hence

$$\begin{aligned} K(N) &\sim \frac{N}{S} \left(\ln \left(\frac{N}{r!tS} \right) + r \ln \ln \left(\frac{N}{r!tS} \right) \right) \\ &\sim \frac{N}{S} \left(\ln \frac{N}{S} - \ln(r!t) + r \ln \ln \frac{N}{S} \right). \end{aligned}$$

(The next term is around $rt! \ln \ln(N/S)/\ln(N/S)$.) This is amazingly sensitive to changes in t , but at least K is infinitely bigger than the K required for $R = 1$. See also the *Perturbations and asymmetry* comment below.

(c) With unbounded $R(N)$, computations are naturally more complicated, but we can see what happens in the special case that $S(N) = \sqrt{N}$. Set $r \equiv r(N) = R(N) + 1$, and set $K = g(N)r\sqrt{N}$; in order for Theorem 1.1 to apply, we require (at least) $g(N) \rightarrow \infty$. We will show that $c = \infty$ (that is, the probability is 0) if $r(N)g(N) \leq \frac{1}{2} \ln N$, and $c > 0$ if $r(N)g(N) \geq (\frac{1}{2} + \eta) \ln N$ for some $\eta > 0$. If $r(N) \geq k \ln N$ for some real $k > 0$, we are out of luck (as $g(N) \rightarrow \infty$ is required for our result, in order that the hypothesis (b), $RN = \mathbf{o}(SK)$, be satisfied).

To compute c , using Sterling's formula, $c = \lim_N f(N)$, where

$$\begin{aligned} f(N) &= \frac{\sqrt{N}(rg)^r \exp(-rg)}{\left(\frac{r}{e}\right)^r \sqrt{2\pi r}}, \quad \text{so} \\ \ln f(N) &= \frac{1}{2} \ln N + r(\ln g - g) + r - \frac{1}{2} \ln(2\pi r). \end{aligned}$$

As $g(N) \rightarrow \infty$, it follows that $rg \sim r(g - \ln g)$.

If $rg \leq \frac{1}{2} \ln N$, then since $r(N) \rightarrow \infty$, we see that $\ln f(N) \rightarrow \infty$, so $c = \infty$. On the other hand, if $r(N)g(N) \geq (\frac{1}{2} + \eta) \ln N$, then $\ln f(N) \leq -\eta \ln N/2$; so $\ln f(N) \rightarrow -\infty$, and thus $c = 0$. •

Perturbations and asymmetry. Asymmetric behaviour appears when we consider perturbations. Here, we permit $R(N)$ to be unbounded (but still subject to all the constraints appearing in Theorem 1.1, e.g., $RN = \mathbf{o}(SK)$). For all sufficiently large N , define $K^0(N)$ to be the solution to $c = 1$, that is (with $r(N) = R(N) - 1$),

$$\frac{(N/S)(SK^0/N)^r \exp(-SK^0/N)}{r!} = 1.$$

Now we perturb K^0 , defining $K(N) = K^0(N) + aN/S(N)$ where a is a real number independent of N ; it can be of either sign. We calculate the new value for c ,

$$\begin{aligned} c_1(N) &= \frac{(N/S)(SK^0/N + a)^r \exp(-SK^0/N + a)}{r!} \\ &= 1 \cdot \left(1 + \frac{aN}{SK^0}\right)^r e^{-a} \\ &\sim \exp(raN/SK^0) \cdot e^{-a} \\ &\rightarrow e^{-a}. \end{aligned}$$

Next, we permit a to vary. If $a \rightarrow \infty$, then $\exp(-c_1) = e^{-e^{-a}} \sim 1 - e^{-a}$ —in other words, convergence (in a) entails exponential convergence (in a) of the probability to 1. On the other hand, if $a \rightarrow -\infty$, then $e^{-c_1} = \exp(-e^{|a|})$. This time, we have *ultra-exponential* decay (in a) to 0. So there is asymmetry in the rates of convergence to one and to zero. •

Significance of conditions a–c. Condition (b) seems very reasonable in the sense that it appears to be close to necessary for the probability to be nonzero (assuming that $K = \mathbf{o}(N)$, which is itself reasonable); however, I have not been able to rule out the possibility that $\alpha(N)$ be bounded (but large), $R(N) \rightarrow \infty$, $K = \mathbf{o}(N)$, and the likelihood of successful occupation being nonzero. The constraints in (a,c) turned up in the course of the proof. It would be interesting to clean up these extra conditions. •

We begin the proof of Theorem 1.1.

For each i , we define $\beta_i(N)$ for the purposes of using inclusion/exclusion, as follows. Pick a collection of i of the N/S intervals, and determine the number of subsets of size $K \equiv K(N)$ that hits each interval in the collection in fewer than $R \equiv R(N)$ elements. Then multiply by $\binom{N/S}{i}$ (the number of choices of i of these intervals. Finally divide by the number of K -element subsets of $1, 2, \dots, N$, that is, divide by $\binom{N}{K}$. Then $1 - \sum_{i=1}^{\infty} (-1)^{i+1} \beta_i(N)$ (if it converges) is the probability, obtained from inclusion/exclusion, that a set of size K will hit all of the disjoint intervals of size $S(N)$, except possibly the last, short, one (if S does not divide N), in at least R points.

As a consequence of Bonferroni's inequalities, there is a unexpected but pleasant convergence property.

LEMMA 1.2 Suppose that $\beta_i(N) \rightarrow b_i$ for all i , and that $\sum_{i=1}^{\infty} (-1)^{i+1} b_i$ converges to C . Then $\sum_{i=1}^{\infty} (-1)^{i+1} \beta_i(N)$ converges (as $N \rightarrow \infty$) to C .

Proof. Set $f(N) = \sum_{i=1}^{\infty} (-1)^{i+1} \beta_i(N)$ (for each N , this is a finite sum). For all k, l , by Bonferroni's inequalities, we have

$$\sum_{i=1}^{2k} (-1)^{i+1} \beta_i(N) \leq f(N) \leq \sum_{i=1}^{2l+1} (-1)^{i+1} \beta_i(N).$$

For each k, l , the sums on the left and right converge as $N \rightarrow \infty$ to $\sum_{i=1}^{2k} (-1)^{i+1} b_i$ and $\sum_{i=1}^{2l+1} (-1)^{i+1} b_i$, respectively. On taking the \liminf on the left and the \limsup on the right, we have that for all k and l ,

$$\sum_{i=1}^{2k} (-1)^{i+1} b_i \leq \liminf_N f(N) \leq \limsup_N f(N) \leq \sum_{i=1}^{2l+1} (-1)^{i+1} b_i.$$

By hypothesis, the left and right sides converge as $k, l \rightarrow \infty$ to the same thing, so $\liminf f(N) = \limsup f(N)$.

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This simplifies the arguments, as it is relatively easy to calculate $b_m := \lim_{N \rightarrow \infty} \beta_m(N)$.

Fix N , $K = K(N)$, $S = S(N)$ and $R = R(N)$, and m . Varying over intervals of the form $[kS, (k+1)S) \cap \mathbf{N}$, we have an exact, but horrendous, formula for the m th term arising from inclusion/exclusion. For each $s = 0, 1, 2, \dots, m \cdot (R-1) - 1$, set

$$D_R(s) \equiv D(s) = \left\{ (i_j) \in \mathbf{Z}_+^m \mid \sum i_j = s; \text{ and } i_j \leq R-1 \text{ for all } j, \right\}.$$

Then

$$\beta_m(N) = \frac{\binom{N/S}{m} \sum_{s=0}^{m \cdot (R-1)} \binom{N-mS}{K-s} \sum_{(i_j) \in D(s)} \prod_{j=1}^m \binom{S}{i_j}}{\binom{N}{K}}.$$

The horrible sum will be considerably simplified—under modest conditions, we can delete all the terms $s = 0, 1, \dots, m(R-1) - 1$, leaving just one term arising from $s = m \cdot (R-1)$, that is, $\left(\binom{S}{R-1} \right)^m \binom{N-mS}{K-m(R-1)}$. This permits us to compute $b_m = \lim \beta_m(N)$ easily.

We wish to obtain reasonable approximations for $\beta_m(N)$. We assume $R(N)^2 = \mathbf{o}(K(N))$ (this, and similar expressions, will usually be abbreviated by dropping the N —thus, $R^2 = \mathbf{o}(K \wedge S)$, $NR = \mathbf{o}(SK)$, $RS \vee RK = \mathbf{o}(N)$, and $S \vee K = \mathbf{o}(N)$).

If f is a polynomial, we use inner product notation, (f, x^s) , to denote the coefficient of x^s appearing in f . Let p the initial segment of degree $R-1$ of $(1+x)^S$. Then $(p^m, x^s) = \sum_{(i_j) \in D(s)} \prod_{j=1}^m \binom{S}{i_j}$. Since $(1+x)^S$ is strongly unimodal (log concave), so is p and thus so is p^m . In particular, the sequence $s \mapsto \sum_{(i_j) \in D(s)} \prod_{j=1}^m \binom{S}{i_j}$ is strongly unimodal. So is the sequence $s \mapsto \binom{N-mS}{K-s}$. Since the Hadamard product of strongly unimodal sequences is itself strongly unimodal, so is

$$(1) \quad s \mapsto \binom{N-mS}{K-s} \cdot \sum_{(i_j) \in D(s)} \prod_{j=1}^m \binom{S}{i_j} := g_N(s).$$

This is over the range $s = 0, 1, \dots, (R-1)m$. In particular, if $g((R-1)m)/g((R-1)m-1) = t > 1$, then $g((R-1)m) < \sum_0^{(R-1)m} g(s) \leq g((R-1)m)/(1-1/t)$ (since the sequence $(g(s))_{s=0}^{(R-1)m}$ is strongly unimodal).

Define $\alpha(N) = SK/RN$. When $s = (R-1)m$, $D(s)$ consists of the one point, $(R-1, R-1, \dots, R-1)$, and when $s = (R-1)m-1$, $D(s)$ consists of m points, $\{(R-1, \dots, R-1, R-2, R-1, \dots, R-1)\}$. Thus it is easy to calculate

$$\begin{aligned} \frac{g((R-1)m)}{g((R-1)m-1)} &= \frac{\left(\binom{S}{R-1}\right)^m \binom{N-mS}{K-(R-1)m}}{\left(\binom{S}{R-1}\right)^{m-1} m \binom{S}{R-2} \binom{N-mS}{K-(R-1)m+1}} \\ &= \frac{(S-R+2) \cdot (K-(R-1)m+1)}{(R-1)m \cdot (N-mS-K+(R-1)m)} \\ &\geq \frac{SK}{NRm} \left(1 - m O\left(\frac{R}{K}\right)\right) \\ &= \frac{\alpha(N)}{m} \left(1 - m O\left(\frac{R}{K}\right)\right). \end{aligned}$$

From (b), $R/K = o(S/N)$; from (c), $S = o(N)$, and it follows that $R = o(K)$. Since m is fixed and $\alpha(N) \rightarrow \infty$ by (b), the expression becomes arbitrarily large. Thus given ϵ , there exists N_0 such that for all $N \geq N_0$, $\sum g(s)/g((R-1)m) < 1 + \epsilon$, that is, almost all the mass of the sum in the definition of $\beta_m(N)$ is concentrated on the last term. [If $R = 1$ or 2 , the corresponding argument is trivial.]

Thus $\beta_m(N)$ is asymptotic (in N) with

$$\beta'_m(N) := \frac{\binom{N/S}{m} \left(\binom{S}{R-1}\right)^m \binom{N-mS}{K-m(R-1)}}{\binom{N}{K}}.$$

Now we can estimate β'_m . The following is elementary.

LEMMA 1.3 If $B < A/2$, then

$$\begin{aligned} \frac{A!}{(A-B)!} &= A^B \exp\left(-\frac{B^2-B}{2A}\right) \left(1 - O\left(\frac{B^3}{A^2}\right)\right) \\ \binom{A}{B} &= \frac{A^B}{B!} \exp\left(-\frac{B^2-B}{2A}\right) \left(1 - O\left(\frac{B^3}{A^2}\right)\right). \end{aligned}$$

Proof. Obviously $A!/(A-B)! = \prod_{i=0}^{B-1} (A-i) = A^B \prod_{i=0}^{B-1} (1-i/A)$. Now take logarithms of the product, and sum. •

The constant(s) that appear in the big Oh terms are slightly bigger than one-half. Obviously, finer estimates can be made, but we won't require them in what follows.

Ignoring some of the $1 \pm O(\cdot)$ terms,

$$\begin{aligned}
\beta'_m(N) &= \frac{(N/S)^m S^{m(R-1)}}{m!((R-1)!)^m} \exp\left(-m^2 \frac{(R-1)^2}{2S}\right) \frac{\prod_{i=0}^{K-1} (K-i)}{\prod_{i=0}^{K-1} (N-i)} \cdot \prod_{i=1}^{K+m(S-R+1)} \frac{N-mS-i}{K-m(R-1)-i} \\
&= \frac{(N/S)^m S^{m(R-1)}}{m!((R-1)!)^m} \left(1 - m^2 O\left(\frac{R^2}{S}\right)\right) \frac{\prod_{i=0}^{m(R-1)-1} (K-i)}{\prod_{i=0}^{mS-1} (N-i)} \cdot \prod_{i=K}^{K+m(S-R+1)} (N-i) \\
&= \frac{(N/S)^m S^{m(R-1)}}{m!((R-1)!)^m} K^{m(R-1)} N^{-mS} N^{m(S-R+1)} \times \\
&\quad \exp\left(-m^2 \frac{(R-1)^2}{2K} + m^2 \frac{S^2}{2N} - m(S-R+1) \frac{K}{N} - m^2 \frac{(S-R+1)^2}{2N}\right) (1 + o(1)) \\
&= \frac{((N/S) \cdot (KS/N)^{R-1} e^{-SK/N} / (R-1)!)^m}{m!} \times \\
&\quad \exp\left(-m^2 \frac{(R-1)^2}{2K} + m^2 \frac{2SR - R^2 - 2S + 2R - 1}{2N} - m(S-R+1) \frac{K}{N}\right) (1 + o(1)) \\
&= \frac{((N/S) \cdot (KS/N)^{R-1} e^{-SK/N} / (R-1)!)^m}{m!} \times \\
&\quad \exp\left(-m \frac{SK}{N}\right) \left(1 + m^2 O\left(\frac{R^2}{K} \vee \frac{SR}{N}\right)\right) \left(1 + m O\left(\frac{RK}{N}\right)\right) (1 + o(1)) \\
&= \frac{((N/S) \cdot (KS/N)^{R-1} e^{-SK/N} / (R-1)!)^m}{m!} (1 \pm o(1))
\end{aligned}$$

To go from the penultimate line to the last line, we observe that $NR = o(SK)$ entails $R^2 = o(SRK/N) = o(K)$ (from (c)). Thus,

$$\begin{aligned}
\beta'_m(N) &= \frac{((N/S) \cdot (R\alpha(N))^{R-1} e^{-R\alpha(N)} / (R-1)!)^m}{m!} (1 \pm o(1)) \\
&= \frac{\left(\frac{N \cdot G_{R-1}(SK/N)}{S \cdot (R-1)!}\right)^m}{m!} (1 \pm o(1)),
\end{aligned}$$

where $G_j(x)$ is function $t \mapsto t^j e^{-t}$, defined for $t > j$.

Hence, if the sequence $((N/S) \cdot G_{R-1}(SK/N) / (R-1)!)$ converges to some number c , then $b_m = c^m / m!$, and by Lemma 1.2, $\sum_{i=1}^{\infty} (-1)^{i+1} \beta_i(N)$ converges to e^{-c} . In this case, the likelihood that a set with cardinality $K(N)$ hits each of the $S(N)$ equi-length intervals in at least $R(N)$ points converges to e^{-c} .

If $((N/S) \cdot G_{R-1}(SK/N) / (R-1)!) \rightarrow \infty$, a routine argument shows the limiting probability is zero, as follows. First, assume $R(N) > 1$ for almost all N . We may select an increasing sequence of positive real numbers, $t_n \rightarrow \infty$ with the property that for all $N \geq n$, $t_n \sqrt{2\pi(R(N)-1)} < N/S(N)$ (possible, since $RS = o(N)$). For $N \geq n$, consider $c(N, n) := t_n(R(N)-1)!S(N)/N$. We have $c(N, n) < (R(N)-1)!/\sqrt{2\pi(R(N)-1)} \leq ((R(N)-1)/e)^{R(N)-1}$. Hence $T_{R(N)-1}(c_{N,n})$ is defined.

For $N \geq n$, set $\alpha_n(N) = T_{R(N)-1}(c_{N,n})/R(N)$. Now fix n , and consider the sequence

$$\left(\frac{N \cdot G_{R(N)-1}(\alpha_n(N) \cdot R(N))}{S(N) \cdot (R(N)-1)!}\right)_{N \geq n}.$$

By construction, this converges to t_n . On setting $K_n(N) = R(N)N\alpha_n(N)/S(N)$, we obtain a new set of parameters, but only the K terms are changed, and the likelihood of successful occupation is e^{-t_n} . Since $T_{R(N)-1}$ is decreasing on its ray of definition, and (t_n) is increasing, we see that $\alpha_n(N) \geq \alpha_{n+1}(N)$ (when $N \geq n$), and so $K_n \geq K_{n+1}$. In particular, $K \leq K_n$ for all n . Hence the likelihood of successful occupation for the original sequence of parameters $(N, S(N), K(N), R(N))$ is bounded above by the corresponding

likelihoods for each of $(N, S(N), K_n(N), R(N))$. Since the latter likelihood is e^{-t_n} and this converges to zero, the likelihood of successful occupation for the original set of parameters is zero.

If $R(N) = 1$ for all N , then the argument is even simpler, since we only have to consider the inverse function of $x \mapsto e^{-x}$.

Ball and bins. Unsurprisingly, the questions just considered are close to, but different from, balls and bins problems. The closest relative, in its simplest form, has n bins and m balls; the balls are tossed randomly into the bins. What is the probability that the minimum number of balls in each bin is at least R ? We will replace m by K . With $N/S = n$, this is different from the preceding problem, in that in the former case, the likelihood that a ball will land in one of the bins (one of the intervals, $[kS, (k+1)S)$) depends on how many are already there. However, it turns out that the asymptotic probabilities are identical.

Balls and bins has a large literature, mostly concerned with optimal strategies, or with the expected maximum number of balls. The basic reference is [RS], although Google Scholar lists well over one thousand papers containing the term *balls and bins* (of course, not all are relevant). I could not find a reference to the expected minimum number of balls, presumably because it is not of interest.

We can easily apply the methods here to the minimal balls and bins problem. Replace m by K , and eventually, n by N/S (to conform with the notation for sets). Suppose we have a set of l bins (corresponding to m in the former context, as in β_m , but it would be too confusing to maintain this convention).

There are $m = K$ balls in $n = S/N$ bins; what is the probability that all bins have at least $r+1 = R$ balls?

Fix $l \leq n$. We obtain an expression for the probability that each the first l bins contain at most r balls, and use inclusion/exclusion as in the previous situation.

Let $(a(1), \dots, a(l); b(l+1), \dots, b(n))$ be a distribution of the m balls (so that $\sum a(i) + \sum b(j) = K$). Then the likelihood of obtaining this distribution of balls is exactly

$$(*) \quad \frac{\binom{K}{a(1), \dots, a(l); b(l+1), \dots, b(n)}}{n^K}.$$

(The total number of distributions, the denominator, is of course given by the sum of the coefficients of $(\sum_{i=1}^n x_i)^K$.) Temporarily fix $a = (a(i))$, the initial segment. We want a formula for the likelihood that a appears as the initial segment of the distribution, that is, the sum over all admissible choices of sequences $(b(l+1), \dots, b(n))$ of the expression appearing in $(*)$ (with a fixed).

Set $s = \sum a(i)$; then $\sum b(j) = K - s$; let B be the set of ordered $n - l$ -tuples of nonnegative integers, $b = (b(l+1), \dots, b(n))$ whose sum is $K - s$. Then

$$\begin{aligned} \sum_{b \in B} \binom{K}{a(1), \dots, a(l); b(l+1), \dots, b(n)} &= \sum_{b \in B} \frac{K!}{\prod a(i)! \prod b(j)!} \\ &= \frac{K!}{\prod a(i)! (K-s)!} \sum_{b \in B} \frac{(K-s)!}{\prod b(j)!} \\ &= \frac{K!}{\prod a(i)! (K-s)!} (n-l)^{K-s} \end{aligned}$$

Hence the probability of obtaining the initial sequence $a = (a(i))$ is (with $s = \sum a(i)$),

$$p(a) := \frac{K!(n-l)^{K-s}}{(K-s)!n^K} \cdot \frac{1}{\prod a(j)!}.$$

Set

$$D(s) = \left\{ a \in \mathbf{Z}_+^l \mid a(i) \leq r \text{ for all } i = 1, 2, \dots, l, \text{ and } \sum a(i) = s \right\}.$$

Thus

$$q(s) := \sum_{a \in D(s)} p(a) = \frac{K!}{\left(\frac{n}{n-l}\right)^K} \cdot (n-l)^{-s} \cdot \frac{1}{(K-s)!} \cdot \sum_{a \in D(s)} \frac{1}{\prod a(j)!}.$$

The functions $s \mapsto (n-l)^{-s}$ and $s \mapsto 1/(K-s)!$ are log concave. Since $\sum_{a \in D(s)} 1/\prod a(i)! = \left((\sum_{t=0}^r x^t/t!)^l, x^s \right)$, the function $s \mapsto \sum_{a \in D(s)} 1/\prod a(i)!$ is also log concave. As the Hadamard product of log concave functions defined on the integers is itself log concave, we obtain that $s \mapsto q(s)$ is log concave.

Now define $\beta_l \equiv \beta_l(N)$ to be the sum over all sets of l bins, of the likelihood that particular set has at most r balls in each bin. This is obviously $\beta_l = \sum_{s=0}^{rl} \binom{n}{l} q(s) = \binom{n}{l} \sum_{s=0}^{rl} q(s)$.

If $q(rl)/q(rl-1)$ exceeds $t > 1$, then from strong unimodularity, we see that $q(rl) \leq \sum q(s) \leq q(rl)/(1-1/t)$. Hence $\sum q(s) = q(rl)(1+O(1/t))$. It is easy to calculate $q(rl)$ and $q(rl-1)$, since $D(rl) = \{(r, r, \dots, r)\}$ and $D(ql-1)$ consists of the l points $\{(r, r, \dots, r, r-1, r, \dots, r)\}$. Thus

$$\begin{aligned} \frac{q(rl)}{q(rl-1)} &= \frac{(n-l)^{rl-1}}{(n-l)^{rl}} \cdot \frac{(K-rl+1)!}{(K-rl)!} \cdot \frac{l((r-1)!)^{l-1}r!}{(r!)^l} \\ &= \frac{K-rl+1}{n-l} \cdot \frac{l}{r}. \end{aligned}$$

We thus assume that (holding l fixed) $K/nr \rightarrow \infty$ (this corresponds to $KN/SR \rightarrow \infty$). (If $r = 0$, a separate, but easier, argument is required.)

Thus $\sum_s q(s) = q(rl)(1+O(nr/K))$, and so (up to multiplication by $1+O(nr/K)$ and with l fixed),

$$\begin{aligned} \beta_l &= \left(\frac{n-l}{n} \right)^K \cdot \prod_{j=0}^{rl-1} (K-j) \cdot \binom{n}{l} \cdot \frac{1}{(r!)^l} \cdot \frac{1}{(n-l)^{rl}} \\ &= \frac{\exp(-Kl/n - Kl^2/2n^2) K^{rl} \exp(-(rl)^2/2K) n^l}{l!(n-l)^{rl}(r!)^l} \left(1 \pm l^3 O \left(\max \left\{ \frac{K}{n^3}, \frac{r^3}{K^2}, \frac{r}{n} \right\} \right) \right) \\ &= \frac{\left(\frac{n(\frac{K}{n})^r \exp(-\frac{K}{n})}{r!} \right)^l}{l!} \left(1 + l^3 O \left(\max \left\{ \frac{K}{n^2}, \frac{r^2}{K}, \frac{r^2}{n} \right\} \right) \right). \end{aligned}$$

Thus, assuming $r^2 = o(K \vee n)$, $K = o(n^2)$, $nr = o(K)$, the formula is exactly the same as in the previous situation (with $n = N/S$). If $\lim nG_r(K/n)/r! = c \in \mathbf{R}^+ \cup \{\infty\}$, then the asymptotic probability is e^{-c} .

Properly stated, we have the following. Recall that $G_r(x) = x^r e^{-x}$.

THEOREM 1.4 Suppose that at discrete time t , $m \equiv m(t)$ balls are thrown at $n \equiv n(t)$ bins. Suppose that $r \equiv r(t)$ is a positive integer-valued function such that

$$\lim_{t \rightarrow \infty} \frac{nG_r(m/n)}{r!} = c \in \mathbf{R}^+ \cup \{\infty\}.$$

Suppose in addition that $r = o(\sqrt{n} \vee \sqrt{m})$, $m = o(n^2)$, and $nr = o(m)$. Then the limit of the likelihoods that every bin contain at least $r(t) + 1$ balls at time t , is e^{-c} .

Motivation. These result were motivated by an (unsuccessful) attempt to show that the Vershik adic map associated to Pascal's triangle is weakly mixing, that is, has trivial spectrum. The idea was that as a path progressed along the triangle, the chances that $e^{2\theta\pi i}$ belong to the spectrum would diminish. However, this depends on the behaviour of the sequence fractional parts, $\{c(m)\theta\}$, where $c(m) = \binom{2m}{m}$ is the central binomial coefficient, for arbitrary irrational θ . This seems impossible to deal with.

Reference

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Mathematics Department, University of Ottawa, Ottawa ON K1N 6N5 Canada; dehsg@uottawa.ca.

Comments and criticisms are appreciated, especially if this turns out to have been done before!